PHOTOMETRIC UNITS AND NOMENCLATURE.

BY EDWARD B. ROSA.

The subject of photometric units and nomenclature has attracted some attention recently in the technical press, and a wish has been expressed by more than one writer that there might come into use a more systematic and uniformly accepted nomenclature. Hering, who has given the subject a good deal of attention and has published some valuable articles on it, remarks that many writers are vague in their expressions, using such terms as intensity, quantity, brightness, illumination, flux, etc., in quite different senses. He says:

Moreover, the application of these useful laws (of light distribution) would be much better understood if we had a clearer physical conception of what these various quantities really mean, instead of merely calling them by indefinite names.

The following discussion is an attempt to bring out the physical meaning of the various quantities referred to, and to show that some of the names of units that have been objected to are really useful and contribute to clear thinking. Many of the theorems derived are not new, but they are nevertheless useful in developing the desired relations between the various photometric quantities. Acknowledgment is made to Blondel, Palaz, Liebenthal, Hering, Kennelly, Sharp, Hyde, Jones and others, whose writings and discussions have done much to develop the subject.

In what follows some of the names are used in a different sense from that ordinarily obtaining, and slight changes have been made in some of the symbols. These changes are in the interest of a more systematic arrangement, and it is hoped they may not be found confusing.

GENERAL DISCUSSION AND DERIVATION OF FORMULAS.

1. *Point source.*—We start with the idea of a luminous flux radiating from a point source. Experiment shows that the illumination produced by such a source varies inversely as the square of the distance from the source. We define the illumina-

tion as the quantity of the luminous flux falling upon a unit of area.

A point source of light of intensity I produces a luminous flux in every direction, the numerical value of which at any given distance is proportional to the intensity I and inversely proportional to the square of the distance. Thus putting E for the *illumination* at a distance r

$$E = \frac{I}{r^2} . \qquad (1)$$

If the light source is at the center of a sphere, the entire inner surface is uniformly illuminated; the light may be said to flow out uniformly in all directions, and the space to be filled with a luminous flux. The total flux falling on the inner surface of the sphere is the product of the illumination, or flux per unit of area, E, times the total area. Putting F for the total *luminous flux* it follows that

$$F = 4\pi r^2 E$$

or, by equation (1)

$$F = 4\pi I. \qquad (2)$$

The intensity I is measured in *candles*,[1] the flux F in *lumens,* and the distance r in centimeters. In practice r is often measured in meters or in feet. Thus from a point source of intensity I *candles,* there is a luminous flux $4\pi I$ *lumens.* This is analogous to the flux of 4π lines of magnetic force from each unit of magnetism, and of 4π lines of electric force from each unit of electricity, in electrostatics.

The *flux density* is the luminous flux per unit of area normal to the flux, or the total flux F over an area divided by the area S; thus the flux density is $\frac{F}{S}$, or $\frac{dF}{dS}$ when it is variable.

If the source is not a point but a small sphere of radius $a,$ the flux $4\pi I$ passes out from a radiant surface $4\pi a^2$. Thus the flux density of radiation or the *specific radiation,* is

$$\frac{F}{S} = \frac{4\pi I}{4\pi a^2} = \frac{I}{a^2} = E'.$$

[1] It is proposed to call the new value of the American candle, which is the same as the English candle and the French bougie decimale, and which is also used by several other countries, the International candle.

Thus, we may speak generally of the luminous radiation at any point in space, and of the flux density of such radiation. If it falls on a material surface the incident flux density is the *illumination* E; as it comes from a luminous or other radiating or diffusing surface, the flux density is the *radiation*, E'. Although E and E' are quantities of the same nature, it is convenient thus to distinguish them, as we shall see farther on.

The luminous flux density in space is analogous to electric displacement in electrostatics, the illumination on a material surface is analogous to surface density of electric charge. We think of an electric displacement as occurring everywhere in space about an electric charge, but a surface density σ occurs only where there is a material conducting body on which the lines of electric force terminate. In the same way the terms luminous flux and flux density apply generally. The *radiation* is the flux density at the source of the flux, and the *illumination* is the flux density or flux per unit of area on the surface where the luminous flux is received.

2. DISTINCTION BETWEEN LUMINOUS FLUX AND ENERGY.

The total luminous flux F is not to be confused with the total energy flowing from a luminous body. Luminous flux, or *light* as we ordinarily say, is the physical stimulus which applied to the retina produces the sensation of light. It is equal to the radiant power multiplied by the stimulus coefficient. This stimulus coefficient is different for every different wave frequency or wave length, and is of course zero for all frequencies outside the visible spectrum. Hence, if W_λ is the power (expressed in watts) for unit of wave length of the spectrum, and K_λ is the stimulus coefficient or *luminous efficiency* whose value varies with the wave length λ, we have for the total power radiated from a body

$$W = \int_o^\infty W_\lambda \, d\lambda$$

and for the luminous flux

$$F = \int_{\lambda_1}^{\lambda_2} K_\lambda \, W_\lambda \, d\lambda$$

where λ_1, and λ_2 are the wave lengths at the limits of the visible spectrum.

As the values of K_λ throughout the spectrum are not accurately known, it is not possible to calculate F in general. But by measuring W in watts and F in lumens, we can determine the ratio of the luminous flux to the radiant power in any particular case. One may properly say that luminous flux is due to and is always associated with radiant power measured in watts: but the statement sometimes made that luminous flux and radiant power can be converted into one another like feet and inches is misleading; for, as stated above, the conversion factor, the stimulus coefficient or luminous efficiency, is not a constant like the ratio of feet to inches, but is variable, having a different value for every different wave length in the visible spectrum and falling to zero outside the visible spectrum.

3. DEFINITION OF INTENSITY.

If the source is not symmetrical, but sends out a total luminous flux F unequally in different directions, then the mean value of the *intensity* is called the *mean spherical intensity*, and its value is

$$I_s = \frac{F}{4\pi} . \tag{3}$$

We thus define the mean spherical intensity with respect to the total flux; and similarly, the intensity I in any particular direction is the ratio of the flux through a small solid angle in that direction to the angle. Thus

$$\left. \begin{array}{l} I = \dfrac{F}{\omega} , \ \omega \text{ being a solid angle,} \\[2ex] \text{or } I = \dfrac{dF}{d\omega} , \ d\omega \text{ being an infinitesimal solid angle.} \end{array} \right\} \tag{4}$$

In the case of a point source or unit sphere radiating equally in all directions, the intensity I is defined as the flux through a unit of solid angle, or steradian, that is, $I = F$ when $\omega = 1$. This is an angle subtended by $\frac{1}{4\pi}$ of a spherical surface, and in the case of a conical angle its section through the apex is a plane angle of 65° 32′ 28″.
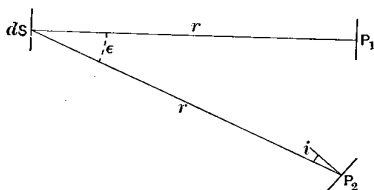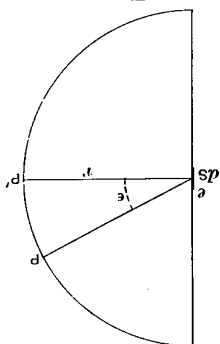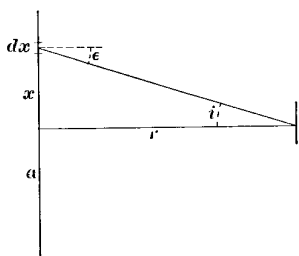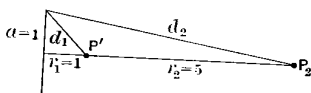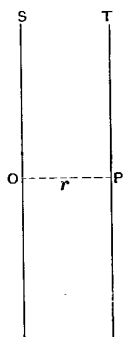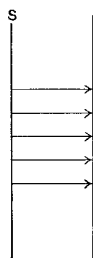
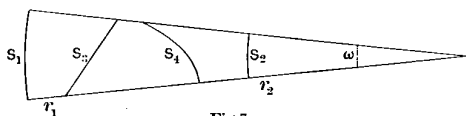Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

4. UNIT DISC.

An elementary disc dS of brightness c gives an illumination at P, Fig. 1, equal to

$$E_p = \frac{cdS \cos \epsilon}{r^2} = \frac{Q \cos \epsilon}{r^2}.$$

Where $Q = cdS$ is the *quantity of light* on the disc.

Integrating this over the hemisphere within which the whole radiation is confined, we have the total flux,

$$F = Q \int_0^{\frac{\pi}{2}} \frac{2\pi r^2 \sin \epsilon \cos \epsilon \, d\epsilon}{r^2}$$

$$= \left[\pi Q \sin^2 \epsilon \right]_0^{\frac{\pi}{2}} = \pi Q = \pi I_1 \qquad (5)$$

Thus, the total luminous flux F from a small plane disc is π times the quantity of light Q on the disc, and also π times the maximum intensity I_1, normal to the disc. The average intensity throughout the hemisphere is one-half of the maximum intensity (πI_1 divided by 2π) and the mean spherical intensity is one-fourth I_1. Thus we have, since $I_s = \frac{1}{4}I_1$.

$$F = \pi I_1 = 4\pi I_s. \qquad (6)$$

That is, the total flux F is 4π times the mean spherical intensity as with a point source or uniform sphere. In the case of the disc, the spherical reduction factor is hence $\frac{1}{4}$. We must therefore carefully distinguish in the various forms of light sources between the mean spherical intensity I_s, the maximum intensity I_1, and the intensity in some particular direction I.

5. EXTENDED SOURCE. CIRCULAR DISC.

Let dS be an element of a plane radiating surface of *specific light intensity* (or brightness) c, defined by the equation

$$I = cdS.$$

That is, the intensity I is equal to the product of c into the small surface dS. Thus, c is the value of the intensity I when the surface is unity, and is the quantity of light per unit of area measured in candles. Thus the intensity I would be measured

by comparing it experimentally with a standard light source, and it is equal to the intensity of a point source or unit sphere which produces the same illumination on a given test screen (of a photometer). Thus, while we *define* the intensity of a light source as the luminous flux per unit solid angle, we *determine* it by comparison with a concrete standard by means of the illumination produced on a test screen at a convenient distance, using a photometer and employing the law of inverse squares.

In Fig. 2 the illumination at P_1 in the normal to dS is

$$E_1 = \frac{ed\mathrm{S}}{r^2},$$

while the illumination at P_2, the angles of emergence and incidence being ϵ and i respectively is

$$E_2 = \frac{ed\mathrm{S} \cos \epsilon \cos i}{r^2}. \tag{7}$$

The cosine law is assumed to hold exactly for both surfaces.

To calculate the illumination due to a circular disc of brightness (*i. e.* specific light intensity) e and radius a on a small plane area P, normal to the axis of the disc at distance r we integrate the effect of each elementary circular ring of the disc. Thus, in equation (7), putting $d\mathrm{S} = 2\pi x dx$,

$$E = e \int_0^a \frac{2\pi x dx \cos \epsilon \cos i}{(r^2 + x^2)}.$$

Since $\cos \epsilon = \cos i = \frac{r}{\sqrt{r^2 + x^2}}$

$$E = \pi e \int_0^a \frac{2x dx \cdot r^2}{(r^2 + x^2)^2} \tag{8'}$$

$$= \pi e \left[-\frac{r^2}{r^2 + x^2} \right]_0^a = \pi e \left[1 - \frac{r^2}{r^2 + a^2} \right]$$

$$\text{or, } E = \frac{\pi e a^2}{r^2 + a^2} = \frac{e\mathrm{S}}{r^2 + a^2} = \frac{Q}{r^2 + a^2}, \tag{8}$$

where Q is the product of the surface of the disc into the specific intensity e, and is the total quantity of light upon the disc meas-

ured in candles. If the disc were very small Q would be the same as the intensity I of the source; but for an extended source we must distinguish between the equivalent *intensity* I and the surface integral of the *specific intensity,* which is Q. The latter we have called the quantity of light upon the disc; it is proportional to the total luminous flux F coming from the extended source, and is equal to F/π, equation (5). Q and F really measure the same thing, except that Q is located on the source and is measured in candles, while F is located in the surrounding space and is measured in lumens; their ratio is constant as $F = \pi Q$ always.[1]

In the case of the disc above mentioned, the illumination E on a small plane normal to the axis is proportional to the total quantity of light Q on the extended source (the circular disc) and inversely proportional to the square of the distance d from P_1 to the *edge of disc.* This holds true for all distances r from zero to infinity. Thus *the law of inverse squares holds generally for* the illumination along its axis due to a *circular disc of any size,* but the distance is measured, not to the center of the disc, *but to the edge.*

Thus we have

$$E = \frac{I}{r^2} \text{ for a point source or a unit disc,}[2]$$

$$\text{and } E = \frac{Q}{d^2} \text{ for an extended disc.} \tag{8a}$$

---

[1] The total quantity of electricity on a disc of area S is equal to the integral of the surface density σ over the area. Thus

$$Q = \int \sigma d S$$

$$= \sigma S \text{ when } \sigma \text{ is uniform.}$$

The brightness or specific light intensity e of a source corresponds to the surface density of electricity σ, and the total quantity of light over a surface is, in the same way, the surface integral of e. Thus

$$Q = \int e d S$$

$$= e S \text{ when } e \text{ is uniform over the area S.}$$

In the case of a sphere, the surface $S = 4\pi a^2$. Therefore, for the spherical source $Q = 4\pi a^2 e$, whereas the intensity $I = \pi a^2 e$. That is, the intensity I of a spherical source is one-fourth of Q, and is equal to the light on a disc of radius a and brightness e. That is, the intensity of the sphere is equivalent to that of a disc of the same diameter, and the same brightness, for points at a great distance.

[2] By unit disc or unit sphere is meant a disc or sphere whose linear dimensions are negligible in comparison with the distance from source to receiver.

To illustrate the rate of variation of the illumination with the distance, let $a = 1$, $r_1 = 1$, $r_2 = 5$.

In the first case for the point P_1, $E_1 = \dfrac{Q}{d_1^2} = \dfrac{\pi e}{2}$.

In the second case for the point P_2, $E_2 = \dfrac{Q}{d_2^2} = \dfrac{\pi e}{26}$.

Thus in the first case the distance is 5 times less and the illumination is 13 times more instead of 25 times more, as it would be if the light Q were all concentrated at the center of the disc. If $r = 0$, the illumination is πe or twice as much as at P_1, and not infinite as it would be at zero distance from a point source.

This theorem is useful in measuring the radiation from walls, as the radiating area may be quite large and the photometer relatively near.

6. INFINITE PLANE.

The radiation from an infinite plane S upon a unit area of a parallel plane T is found by integrating equation (8') to infinity. Thus

$$E = \pi e \int_0^\infty \frac{2x\,dx \cdot r^2}{(r^2 + x^2)^2} = \pi e \left[\frac{-r^2}{r^2 + x^2} \right]_0^\infty = \pi e. \qquad (9)$$

Thus the flux density or *illumination* at any point P on the T plane is π times the brightness or specific light intensity e on the radiating plane S, and is independent of the distance r.

From each unit of area of S having a specific light intensity e, the total flux is πe, as shown in (5) above. The resultant flux at all points is the same as though the total flux πe from each unit of area of S was confined to a cylindrical tube of unit area perpendicular to S, in which case the flux density would of course be constant at all sections, that is at all distances.

7. INFINITE CYLINDER.

In a similar manner we may consider the flux from an infinite circular cylinder of uniform specific intensity e, and radius a.

The flux coming from unit length of the cylinder is πe times the area. Hence $F = 2\pi^2 a e$; whereas the flux falling on the

inner surface of a concentric cylinder of radius r, is E times the area, E being the illumination. Hence, for a unit of length of the cylinder $F = 2\pi r E$. Therefore,

$$E = \frac{\pi a e}{r} = \tfrac{1}{2}\frac{Q}{r} . \tag{10}$$

Thus the illumination due to an infinite cylinder varies *inversely as the distance*. This is intermediate between the case of the point source, for which E is inversely as r^2, and the infinite plane, where E is independent of the distance; that is, proportional to r^0.

The quantity Q for the luminous cylinder is e times the surface. Therefore the *quantity per unit of length* is

$$Q_1 = 2\pi a e \tag{11}$$

The total luminous flux F, as stated above, is $2\pi^2 a e$. Hence the total flux per unit of length F_1, is π times the quantity or

$$F_1 = \pi Q_1.$$

or, for any portion (or the whole) of an infinite cylinder of uniform specific intensity, the total flux is π times the quantity; that is,

$$F = \pi Q \tag{12}$$

as shown above for a circular disc.

8. UNIT LENGTH OF CYLINDER.

Suppose a light source in the form of a very long cylinder of radius a and uniform specific intensity e. It is desired to determine experimentally its total luminous flux F. Suppose one has measured by means of a photometer the equivalent intensity I_1 of unit length of the cylinder, (screening the photometer from all but a short section of the cylinder): we are to calculate the total flux F from I_1. The unit length of cylinder will produce the same illumination at a distance as a rectangular plane of breadth $2a$ and height unity of specific intensity e equal to that of the surface of the cylinder. Hence the equivalent intensity I_1 is equal to $2ae$ and the illumination produced on a photometer screen at distance r is
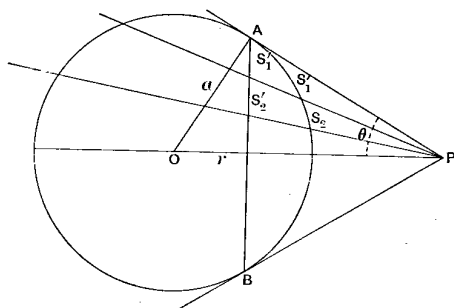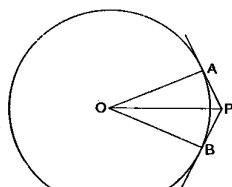
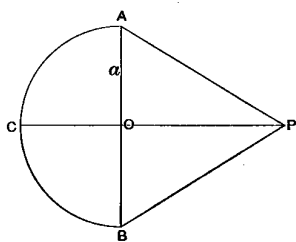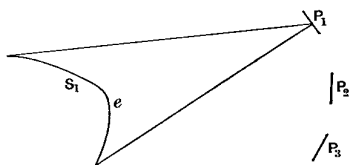$$E = \frac{2ae}{r^2} = \frac{I_1}{r^2} .$$
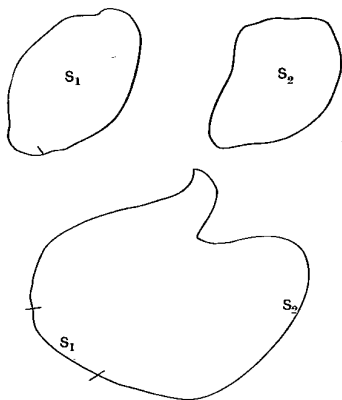
Fig. 8

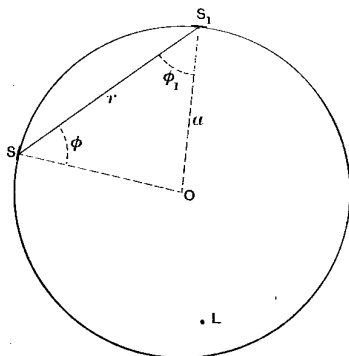Fig. 9

Fig.10

Fig.11

Fig.12

Fig. 13

The quantity of light on the cylinder per unit of length is e times the surface or $2\pi ae$; and the total flux F_1 is π times the quantity.

Thus we have

$$F_1 = 2\pi^2 ae$$
$$I_1 = 2ae$$
$$\therefore \quad F_1 = \pi^2 I_1 \qquad (1.)$$

Thus to obtain the total luminous flux F_1 from the measured value of the equivalent intensity of a unit of length of the luminous cylinder we multiply this intensity I_1 by π^2, instead of multiplying by 4π as in the case of a sphere.

The spherical reduction factor of a short cylinder (the convex surface only being luminous) is therefore $\pi^2/4\pi = \pi/4 = 0.785$. This would be nearly true for an incandescent lamp having a single, straight filament. The value for a hairpin filament would be only slightly larger.

If the cylinder is quite long, we should then get the total flux F by multiplying F_1 by the length of the cylinder. This demonstration is of course based on the assumption that the cosine law holds for the cylinder. If the source is a long tube, like the Moore light, the result would be subject to any modification dependent on its departure from the cosine law.

Thus while the total flux F is always π times the quantity Q of the source, it is not always 4π times the intensity. It is 4π times the intensity I for a point source or sphere, $\pi^2 L$ times the equivalent intensity I_1 (measured at a relatively great distance) of unit length of a long cylinder, L being the length, and πS times the equivalent intensity I_1 of unit of area of a plane, S being the area of the plane.

It is, however, always 4π times the *mean spherical intensity* of the given source. The illumination produced by a short cylinder is approximately inversely proportional to the square of the distance. For all distances greater than five times the length, the departures are not greater than 0.2 per cent. in a particular case worked out by Hyde; the diameter of the cylinder in this case was one-tenth the length. The exact expression for the illumination due to a finite cylinder is not simple, and the calculation tedious.

9. CASE OF LARGE SPHERE.

If a surface S_1 (suppose a portion of a spherical surface of radius r_1) has a specific light intensity (brightness) e and subtends a small solid angle ω, the illumination which it produces at P is

$$E = \frac{eS_1}{r_1^2} = \frac{e\omega r_1^2}{r_1^2} = e\omega . \qquad (14)$$

A second surface S_2 of the same specific intensity will produce the same illumination at P provided it subtends the same angle ω. A third surface, S_3, at any angle will also produce the same illumination at P if it has the same specific intensity e and subtends the same solid angle ω. For the radiation of each element dS_3 is

$$\frac{edS_3}{r_3^2} \cos \epsilon = \frac{e\omega r_3^2}{r_3^2} = e\omega$$

as before. So also with the curved surface S_4. In every case the greater distance from P or the inclination of the angular position is compensated by the greater area included within the given solid angle.

Let us calculate the illumination at P due to a large luminous sphere of radius a and specific intensity e, r being the distance from P to the center of the sphere. Let the solid angle APB subtended at P by the sphere be subdivided into a large number of elementary solid angles. Each of the latter encloses an area, as S_1, on the surface of the sphere, and also a corresponding area S_1', on the circular disc AB. As we have just seen, the illumination produced at P by each spherical area S_1, S_2, etc., is exactly the same as that produced by the corresponding plane areas S_1', S_2', etc., of the disc, if the specific light intensity e is the same for the disc as for the sphere. Therefore, the illumination at P due to the entire sphere is the same as that due to the disc AB, and we can calculate the latter by formula (8a). That is,

$$E = \frac{Q}{d^2} ,$$

where Q is the quantity of light on the disc and d is the distance

AP from the point P to the edge of the disc. Q is equal to e times the area of the disc, or

$$Q = \pi(a \cos \theta)^2 \cdot e$$
$$d = r \cos \theta$$
$$\therefore E = \frac{Q}{d^2} = \frac{\pi a^2 e}{r^2}$$
$$= \frac{1}{4}\frac{Q_s}{r^2} = \frac{I_s}{r^2} , \qquad (15)$$

where Q_s is the quantity of light on the sphere $= 4\pi a^2 e$ and is constant for all distances, and I_s is the intensity of the equivalent point source. Therefore, the illumination produced by a *sphere of any size* is inversely proportional to the square of the distance measured *from its center,* and is equal to the intensity of a point source (or unit sphere) having the same total amount of light divided by the square of the distance. In other words, the inverse square law holds just as rigorously for large spheres as for points, (always, of course, assuming the cosine law to hold for the spherical surfaces, and the specific intensity e to be uniform over the sphere). When P comes very near to the surface the area AB of the sphere available for illuminating P is very small, but the distance is just enough less to counterbalance When P comes up to the surface, $r = a$, and

$$E = \pi e$$

the same as for an infinite plane, to which the sphere is equivalent when the distance from the surface is reduced to zero.

The same result is reached more simply as follows:

A luminous sphere of radius a and uniform specific light intensity e gives off a total flux $F = 4\pi a^2 \times \pi e = 4\pi^2 a^2 e$. This produces an illumination on the inner surface of any concentric sphere, which by symmetry will be uniform over any spherical surface and $F = 4\pi r^2 E$.

$$\therefore E = \frac{\pi a^2 e}{r^2} = \frac{I}{r^2} .$$

Therefore, *the illumination produced by a sphere of uniform specific intensity e is inversely proportional to the square of the distance from the center for all distances, from the surface of the sphere to infinity.*

10. RECIPROCAL RELATIONS.

From what precedes we see that the illumination at any point P due to the hollow hemisphere ACB is the same as that due to the circular disc AOB. The latter is

$$E = \frac{\pi a^2 e}{AP^2} . \qquad (16)$$

When OP is reduced to zero, the illumination due to the disc is πe, and hence the illumination at O on an elementary plane area in the diametrical plane is π times the specific intensity e of the surface of the sphere. We have already seen that the total flux from a unit of surface of intensity e is πe. Hence the total flux through unit area S at O, due to the hemisphere, is equal to the total flux through the hemisphere due to the luminous unit area S, the specific intensity e being the same in each case.

This is a particular case of a more general proposition, namely; *the flux due to any surface S passing through an element dS is equal to the flux due to the latter passing through the former,* the specific intensity being the same in each case.

As shown above, the illumination E at P_1 due to S_1, is equal to $e\omega$ where e is the specific intensity of S_1 and ω is the solid angle subtended at P_1 by S_1, this is independent of the shape of S_1 or its distance from P_1. The flux F passing through dS at P_1 is therefore

$$F = \int e d\omega dS \cos \theta, \text{ over the area of } S_1. \quad \text{Or}$$

$$F = edS \int d\omega \cos \theta. \qquad (17)$$

Similarly, the flux due to dS at P_1 passing through S_1 is

$$F = \int e dS \cos \theta \, d\omega$$

$$edS \int \cos \theta \, d\omega.$$

In the integration, every element $d\omega$ of the solid angle is to be multiplied by the cosine of the angle it makes with the normal to the area dS.

As the same theorem holds for the elementary areas P_2 and P_3, etc., it holds for their sum, and hence for a finite surface S_2.

Hence we see generally that *the luminous flux due to a surface S_1 passing through S_2 is equal to the luminous flux due to S_2 passing through Sr*, the specific intensities e being the same in each case. This is analogous to the theorem that the magnetic flux due to a magnetic shell S_1 which passed through a second shell S_2, is equal to that part of the magnetic flux of S_2 which passes through S_1, the strength of the shells being supposed the same. Or, again, the number of lines of force due to unit current in an electric circuit S_1 passing through S_2 is equal to the number of lines of force due to unit current in S_2 passing through S_1. It follows from the above that in any closed surface of uniform specific intensity e, the flux passing out from any portion S_1 is equal to that received from the remainder of the surface, S_2.

II. HOLLOW SPHERE.

Suppose a hollow sphere of uniform surface having a coefficient of diffuse reflection m.

$$1 - m = \text{absorption.}$$
$$\text{Let } E = \text{illumination at S.}$$
$$E' = mE = \text{radiation from S.}$$
$$e = \frac{mE}{\pi} = \text{specific intensity or brightness of S.}$$

The flux falling on S_1 due to S is,

$$S_1 dE_1 = \frac{eSS_1 \cos^2 \phi}{r^2} = \frac{mE}{\pi} \frac{SS_1 \cos^2 \phi}{r^2}. \tag{18}$$

$$\text{But } r = 2a \cos \phi$$
$$r^2 = 4a^2 \cos^2 \phi \qquad \therefore dE_1 = \frac{mE}{\pi} \frac{S}{4a^2}.$$
$$\frac{\cos^2 \phi}{r^2} = \frac{1}{4a^2}$$

and this is the same for every element of the sphere. Hence *every element* illuminates all other elements equally. Therefore, the *indirect* illumination of the sphere must be the same everywhere, no matter how unequal the direct illumination may be. That is, a light at L illuminates the sphere unequally, directly. But that part of the total illumination due to diffuse reflection is, notwithstanding, equal everywhere.

A light of mean spherical intensity I sends out $4\pi I$ lumens.

Of this there is reflected, 1st, $4\pi mI$ lumens.

Of this there is reflected, 2d, $4\pi m^2 I$ lumens.

Of this there is reflected, 3d, $4\pi m^3 I$ lumens.

Therefore, total amount of flux reflected is $4\pi Im \left[1 + m + m^2 + m^3 + \cdots \right] = 4\pi I \dfrac{m}{1-m} = F_2.$

Hence, the secondary illumination, everywhere equal on the surface of the sphere, is

$$E_2 = \frac{F_2}{4\pi a^2} = \frac{mI}{a^2(1-m)} \qquad (19)$$

Thus, the indirect illumination is proportional to I, and the lamp of intensity I may be anywhere in the sphere. It is equal to $\dfrac{m}{1-m}$ of what the direct illumination would be if the source were placed at the centre of the sphere.

For example, let a 16 candle-power lamp be placed within a sphere having a radius of one meter, and a coefficient of diffuse reflection of 0.8.

Then I = 16.

a = 1 meter.

m = 0.8.

$E_2 = \dfrac{0.8}{0.2} \dfrac{16}{1} = 64$ meter candles.

$E_1 = \dfrac{I}{a^2} = 16$ meter candles, if lamp is in the center.

$E = E_1 + E_2 = 80.$

Thus, the total illumination is five times what it would be if the walls were perfectly black. We can put this in another way: Of the total illumination of 80 meter candles, 20 per cent. is absorbed by the walls. Therefore, the lamps or source must supply only one-fifth of the total illumination, just enough to make good the constant loss.

Thus, the source is analogous to an exciter of electric waves that must supply just enough energy to make good the friction or resistance losses in the circuit.

12. LUMINOUS FLUX WITHIN AN ENCLOSURE.

If the inner surface of the hollow sphere has a brightness e, and a specific radiation $E' = \pi e$, a unit disc at the center of the sphere will receive an illumination $E = \pi e$. The same will be true wherever the unit disc is placed within the sphere, and whatever the orientation of the disc. That is, the flux falling on the disc will be everywhere the same. The flux density within the hollow sphere is therefore everywhere uniform and equal to πe. The flux from a point source is thought of as in straight lines, and a disc can be placed normal to the direction of the flux. But within the sphere the flux has a uniform value, but no resultant direction.

Within a cube or enclosure of any shape, of which the walls have a uniform brightness e or uniform specific radiation E' the same condition obtains as in the sphere; namely, the luminous flux is everywhere the same, and a small area will have the same illumination no matter where it is placed or how it is oriented. This is seen by dividing up the space about any point P into elementary solid angles. The illumination due to the surface subtending an angle ω is independent of the distance from P, and hence it will be πe for the total angle 2π on either side of the surface at P, no matter where the surface is placed.

The same is true therefore for the space between two infinite planes of brightness e. The illumination is πe on a small plane at P_1, P_2 or P_3, anywhere between the two radiating planes S and T no matter how they may be placed. Evidently we cannot think of the flux as normal to the planes, as with the lines of force due to electrostatic charges on the planes S and T. The luminous flux normal to P_3 is the same as normal to P_1. On the other hand, the electric force normal to P_3 would be zero.

These theorems have a practical application in the lighting of rooms.

13. SUMMARY OF PHOTOMETRIC RELATIONS.

The preceding discussion has shown the necessity for distinguishing several different photometric quantities which are sometimes confused. One writer has advocated the use of the fewest possible names, and has tried to show that *intensity, flux* and *flux*

density are sufficient. In order to fix our ideas more clearly it will be advantageous now to state as concisely as possible the definitions of the several quantities and distinctions between them.

Luminous flux, or *light* as the term is used in photometry, is the usual physical stimulus which excites vision. It is propagated by means of the vibratory motion in the ether, and the frequency of the vibrations, or the combination of frequencies present in any given case, determines the color. The total quantity of flux F flowing away from a monochromatic luminous source is
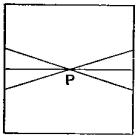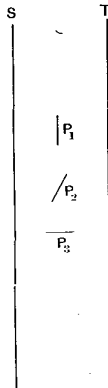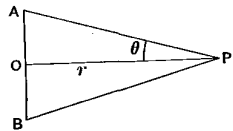

Fig.14   Fig.15   Fig.16

proportional to the total radiant energy, and to a stimulus co-efficient, the latter being the luminous efficiency K_λ for the particular frequency or wave length of the given radiation. Thus the equations

$$F = K_\lambda W$$
$$K_\lambda = \frac{F_\lambda}{W_\lambda}$$

express the luminous flux as the power W multiplied by the luminous efficiency K_λ, and if flux is expressed in *lumens* and the power in *watts*, the luminous efficiency is the number of lumens per watt of radiation of the wave length λ. For white or chromatic light, K will have a value depending on the distri-

bution of the energy in the spectrum. It is a maximum in the yellow green region and falls off rapidly in either direction, reaching zero at the limits of the visible spectrum. The luminous efficiency of most light sources is greatly reduced by the amount of radiation outside the visible spectrum, chiefly of longer wave length than that of visible radiation, and the total efficiency of such a source,

$$K = \frac{F}{W}$$

is the quotient of the total luminous flux divided by the total radiant power.

For the purposes of definition and of expressing the mathematical relations involved in photometry, it is permissible to confine ourselves to monochromatic light, and to consider K a constant, although it does in fact vary somewhat with the magnitude of the flux density. We also assume that all surfaces are perfectly diffusing and obey the cosine law, and that there is no absorption in the atmosphere.

The intensity of a point source or uniform luminous sphere is measured by the luminous flux flowing through a unit solid angle whose apex is the given point or center of the given sphere. Thus from a source of intensity I, light is flowing away at a rate of I lumens per unit solid angle or a total of $4\pi I$ lumens for the point source or uniform sphere. If the source is not uniform, and light is flowing away at unequal rates in different directions, the intensity I in any direction is equal to the flux dF in an elementary solid angle $d\omega$ taken in the given direction. Thus

$$I = \frac{dF}{d\omega}$$

is a general expression applying to all point sources whether radiating equally or unequally in different directions. If the unsymmetrical source is extended, as for example an incandescent lamp or a diffusing globe, the same holds true if the distance at which the measurements are made are sufficiently great so that the distribution of light is practically the same as from an unsymmetrical point source. For less destances than this, the intensity is not a constant in a given direction, but varies with r.

In this case the equivalent intensity at any point is equal to that of a point source which gives the same flux density, or *lumens* per sq. cm., at the point that the given source does. The mean spherical intensity I_s is the average value of the intensity and is equal to the total flux F divided by 4π.

The total flux from a given extended source is therefore a constant independent of distance, as is also the mean spherical intensity I_s. The intensity I in a particular direction, however, in the case of extended sources other than spheres varies with the distance, but at relatively great distances the variation is inappreciable.

Thus the luminous flux is the fundamental quantity. But while we *define* I as the flux per unit solid angle, or *rate of flux with respect to solid angle,* we *determine* I by comparison with a concrete standard. Thus photometric standards are really standards of light flux, their values being expressed in *candles.*

If f is the spherical reduction factor with respect to any particular direction, and I is the intensity of a source in that direction,

$$I_s = fI$$

For a unit disc, that is a small circular disc of uniform brightness, the total flux is π times the normal intensity I_1, whereas the mean spherical reduction factor with respect to the normal is $\frac{1}{4}$. Hence, the total flux is

$$F = \pi I_1$$
$$= 4\pi I_s, \text{ as for a sphere.}$$

In general, for any light source, $F_s = 4\pi I_s = 4\pi f I$, but for extended sources other than spheres, the value of f as well as I varies with the distance from the source for points relatively near the source.

The *specific flux* or *flux density* is the luminous flux per unit of area, or lumens per square centimeter. When the flux falls upon a material surface, we call the specific flux the *illumination,* E. When we speak of the flux coming *from* a surface, whether it be a self-luminous source at high temperature, or a reflecting or radiating surface at low temperature, we call the specific flux the specific radiation, or simply the *radiation,* E'.

Thus the illumination E is

$$E = \frac{F_i}{S} = \frac{dF_i}{dS} = \frac{I}{r^2}.$$

The radiation E' is

$$E' = \frac{F_e}{S} = \frac{dF_e}{dS}.$$

F_i is the incident flux, F_e is the emitted or radiated flux. If m is the coefficient of diffuse reflection or transmission, $(1 - m)$ being the absorption,

$$F_e = mF_i$$
$$E' = mE,$$

where the radiation consists in the diffuse reflection or transmission of a portion of the incident flux or illumination.

The radiation or illumination when large may be expressed in lumens per sq. cm.; when small in milli-lumens per sq. cm. The milli-lumen per sq. cm. is nearly equivalent to the foot-candle.

1 lumen per sq. cm. = 10,000 lumens per sq. meter.
= 10,000 meter candles.
1 milli-lumen per sq. cm. = 10 meter candles = 10 lux.

$$= \frac{1}{1.0765} \text{ foot-candles.}$$

Specific intensity e of a source is the intensity in candles per sq. cm. of area, taken normally. Thus

$$e = \frac{I}{S} = \frac{dI}{dS}.$$

Brightness, or *specific* light intensity, refers to the quantity of light per unit of area of a source, and is measured in candles per sq. cm. Brightness can refer equally to luminous sources of relatively high specific intensity or to reflecting and radiating sources of low intensities. The latter may be conveniently expressed in milli-lumens per sq. cm. Thus we may say a flame has a specific radiation of 10 lumens per sq. cm. or a brightness (specific intensity) of 0.8 candles per sq. cm.; and a wall has a specific radiation of 10 milli-lumens per sq. cm., or a brightness of 0.8 milli-candles per sq. cm. or of 8 candles per sq. meter.

The quantity Q is proportional to the total amount of light

emitted by the source, and is equal to the surface integral of the specific intensity e. Thus

$$Q = \int e d S.$$

The quantity for a small luminous circular disc of radius a and uniform specific intensity e is

$$Q = \pi a^2 e = I_1.$$

That is, the quantity is equal to the maximum intensity. In this case, the whole surface is equally effective in producing the illumination on the test screen by which the intensity I is measured. But for an extended disc, the quantity and the normal intensity, as we have seen above, are not the same. Thus, the quantity is e times the surface, or

$$Q = \pi a^2 e,$$
$$E_1 = \frac{Q}{a^2 + r^2} = \frac{I_1}{r^2},$$
$$\therefore \frac{I_1}{Q} = \frac{r^2}{a^2 + r^2} = \frac{r^2}{d^2} = \cos^2 \theta.$$

That is, the normal equivalent intensity of the disc with respect to the point P is Q times $\cos^2\theta$. When the distance is equal to the radius of the disc, the quantity Q is twice the normal intensity I_1.

The total luminous flux is $\pi e S$ or π times the quantity, and the mean hemispherical intensity is $\frac{Q}{2}$ or half the quantity.

In the case of a sphere of uniform specific intensity e the quantity is $\int e d S = 4\pi a^2 e$. The intensity $I = \pi a^2 e$. Hence the intensity is one-fourth the quantity. In other words, the total radiation from the sphere is four times as great as from a unit disc of the same normal intensity. The relation between quantity and intensity for a few simple cases are

For a unit disc $I_1 = Q$.

For an extended circular disc $I_1 = Q \cos^2 \theta = Q\frac{r^2}{d^2}$.

For a sphere $I = \frac{1}{4}Q$.

For a unit cylinder $I = \frac{1}{\pi} Q$.

The total luminous flux delivered in a given time, that is the time integral of the luminous flux, may be expressed in lumen seconds or lumen hours, according to circumstances. Thus, putting L for the *total lighting* in the time T

$$L = FT$$
$$= \int F d'T, \text{ if F is variable.}$$

where F is in lumens and the time is expressed in the most convenient unit. The flash of a firefly may be expressed in lumen-seconds; the quantity of light per gram of an illuminant, or the total light given during the life of an incandescent lamps is better expressed in lumen-hours.

Since flux of light may also be expressed in spherical candles, $\left(\frac{1}{4\pi} \text{ times the lumens}\right)$ we may also express the time integral or total quantity of light in terms of spherical candles and hours. Thus

$$L = I_s T$$
$$= \int I d'T, \text{ if the spherical candle-}$$

power is a variable with respect to T, the value of L being here given in candle-hours.

The photometric quantities employed in the preceding discussion are shown in Table I, together with the units in which are expressed and the equations of definition.

The symbol F has been employed for the flux (as originally proposed by Hospitalier) instead of Φ for the following reasons:

1. Φ is the only Greek letter in the series, and it is more consistent to use a Latin letter; F is the initial letter of the word flux.

2. The letter Φ is more or less unfamiliar to many illuminating engineers, and also to many printing offices, and it is often confused with the small letter ϕ which is used for an angle.

The symbol E' is used for radiation instead of R (as proposed by Hospitalier) because it is so closely related to the illumination. Blondel and others proposed to employ the same letter E for illumination and radiation, but that gives rise to confusion.

TABLE I.

Photometric magnitude.	Symbol.	Unit.	Equation of definition.
1. Intensity of Light.........	I	Candle	$I = \dfrac{F}{\omega}$
2. Luminous Flux	F	Lumen	$F = I\omega = \dfrac{IS}{r^2} = ES = \pi Q$
3 Illumination	E	$\dfrac{\text{Lumens}}{m^2}$ or $\dfrac{\text{milli-lumens}}{cm^2}$	$E = \dfrac{F_i}{S} = \dfrac{I}{r^2}$
4. Radiation................	E'	Lux = meter-candle	$E' = \dfrac{F_e}{S} = \pi e = mE$
5. Brightness	e	$\dfrac{\text{Candles}}{cm^2}$	$e = \dfrac{I}{S \cos \epsilon}$
6. Quantity................	Q	Candles	$Q = eS$
7. Lighting.................	L	Lumen-hours	$L = FT$

I, e, Q are expressed in candles.   F, E, E' are expressed in lumens.   L is in lumens or spherical candles.

$E' = \pi e.$ $F = \pi Q.$ F_i = incident flux. F_e = emergent flux.

m = coefficient of diffuse reflection or transmission $(1 - m)$ = coefficient of absorption.

On the other hand, E′ gives sufficient distinction, and at the same time recalls their close connection. The letter e is used instead of i for the *specific intensity* or brightness because it has been used in France and Germany, and hence its use in the English language tends to uniformity. Quantity of light, Q is here used as the surface integral of e instead of the time integral of F. It is analogous to quantity of electricity in electrostatics, and is more properly employed in the sense here used than with the other meaning: The term *lighting* for *flux* times *time* is used in harmony with the usage in France and Germany.

III. PROBLEMS FOR ILLUSTRATION.

Problem 1.—A lamp of 200 candle-power (supposed uniform in all directions) is placed in the center of a spherical diffusing globe of 40 cm. diameter, the absorption of which is 30 per cent. Required, the intensity of the globe, its specific intensity, its specific radiation, the illumination on its inner surface, and the illumination it produces at a distance of 3 meters from the center of the globe.

The illumination on its inner surface is

$$E = \frac{I}{a^2} = \frac{200}{400} = 0.5$$

lumens per sq. cm., (formula 1). The radiation E′ is mE, where m is one minus the absorption; it is here 0.7. Therefore, the radiation is 0.35 lumens per sq. cm. The specific intensity e is $\dfrac{E'}{\pi}$ or 0.112 candles per sq. cm. The intensity I of the globe is $200 \times 0.7 = 140$ candles. The illumination E at a distance of 3 meters

$$E = \frac{140}{300^2} = 0.00156 \text{ lumens per sq. cm.}$$
$$= 1.56 \text{ milli-lumens per sq. cm.}$$
$$= 1.45 \text{ foot-candles.}$$
$$\text{or } E = \frac{140}{3^2} = 15.6 \text{ meter-candles.}$$
$$= 15.6 \text{ lux.}$$

Problem 2.—A circular area S, two meters in diameter, on the side of a wall is uniformly illuminated, E being 4 meter-candles.

A photometer placed one meter from the wall, perpendicular to the center of the illuminated area, measures the equivalent intensity I of the area S, and finds it to be 1 candle. What is the absorption coefficient of the wall?

The illumination E being 4 meter-candles, and the area S being π square meters, the flux F falling on the area S is 4π lumens. The measured intensity I at a distance $r = 1$ meter is 1 candle. Therefore, the quantity of light on the disc is

$$Q = I_1 \frac{d^2}{r^2} = 1 \times \frac{2}{1} = 2 \text{ candles.}$$

The total flux from the disc is π times the quantity Q. Therefore, the total flux coming from the area S is 2π lumens, whereas the flux falling upon it is 4π lumens. Therefore, the coefficient of absorption is ½ or 50 per cent.

Problem 3.—Suppose a room of 900 square meters total wall surface is to be so lighted that the walls shall have an average illumination of 10 lumens per square meter, the coefficient of absorption of the walls being 40 per cent on the average. How many lamps of 15 mean horizontal candle-power will be required?

Part of the illumination will be due to light reflected from the walls. The lamps must supply that which is absorbed. The flux to be supplied is therefore $F = 0.40 \times 900 \times 10 = 3,600$ lumens. If each lamp has a spherical radiation factor of 80 per cent., it will supply $4\pi \times 0.80 \times 15 = 150$ lumens, approximately. Hence, 24 lamps will be required.

(Examples 1 and 3 are borrowed from one of Blondel's papers.)

COLLECTION OF FORMULAS.

1. $E = \dfrac{I}{r^2}$ for point source, unit sphere of any size. $I = 4\pi a^2 e$, where $a =$ radius of sphere and $e =$ brightness of surface.

2. $E = \dfrac{\pi a^2 e}{r^2}$ for sphere of radius a.

   $= \pi e$ when $r = a$; that is, at surface of sphere, same as for an infinite plane.

3. $E = \dfrac{\pi a^2 e}{a^2 + r^2} = \dfrac{Q}{d^2}$ for disc of radius a, at distance r on axis.

   $d =$ distance of point on axis to edge of disc.

4. $E = \dfrac{\pi a e}{r}$ for infinite cylinder, $e =$ brightness or specific light intensity. $a =$ radius

   $= \dfrac{Q_1}{2r}$, where $Q_1 =$ quantity of light per unit of length

   $= \pi e$ at surface.

   $I_1 = \dfrac{2ae}{r^2} =$ intensity per unit of length.

5. $E = \pi e$ for infinite plane, *at all distances*.

6. $E = \dfrac{edS \cos \epsilon}{r^2} = ed\omega$ for any small surface dS subtending a small angle $d\omega$ at any distance.

7. $E = \dfrac{2e}{r} \cos \epsilon$, for infinitely long, very narrow strip of e units of light per unit of length

   $= \dfrac{2Q_1}{r}$.

8. $Q = \displaystyle\int edS$ over sphere, cylinder, disc or other surface where $e =$ normal intensity.

9. $F_s = \pi Q$, for sphere or other extended source.

10. $E = \dfrac{Q}{d^2} = \dfrac{I_o}{r^2} \therefore \dfrac{I_o}{Q} = \dfrac{r^2}{d^2} = \cos^2 \theta$, for a disc.

   $I_o =$ equivalent point source, $Q =$ quantity of light over disc

   $I = Q/4$ for a sphere.