The Calculation of Illumination from Sun and Sky

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Formulas are here given which express the connection between the height of the sun above the horizon and the illumination from sun and sky on a horizontal surface. With the aid of these formulas the average illumination on the ground can be calculated for any day in the year and for any hour whatever.

Introduction

IT IS perhaps supposed that the daylight, that is the light from sun and sky, is so variable that it would be impossible to calculate its intensity. The daylight, of course, changes every hour in the day; in the winter it is less intense than in the summer, and clouds of all kinds reduce it in an apparently incalculable way. Moreover, it varies from place to place; in a factory town, with a smoky and vapory atmosphere, it is different than in the country, with a clear and translucent atmosphere.

Nor is it possible to calculate the exact intensity of the sunlight at any given moment. It must be measured with suitable instruments. What, on the other hand, can be calculated is the average illumination at the time and place in question, and this is often of greater value in illumination studies of different kinds than the results obtained from single measurements. Such calculations can be made if we know the laws according to which the light from sun and sky changes.

Illumination from the Sun

The intensity of the sunlight is, on a preliminary view, a function of the time of year (date) and the time of day (hour). But these data merely determine the height of the sun above the horizon, and it is this which mainly determines the intensity of the solar radiation. True, that the time of year has some influence in itself; thus, for example, the sun in winter is somewhat nearer the earth than in the summer, seeing that the earth's orbit is elliptic, and therefore sends somewhat more energy towards the earth. Furthermore, the air in winter is usually clearer and drier, whereas in the late summer it is filled to a greater extent with

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fumes and vapor. Such variations according to the season, however, are not so marked as to prevent the adoption of an average for the intensity of illumination at different solar altitudes.

By basing the calculation of the intensity on the height of the sun, instead of on the date and hour, several practical advantages will be gained, such as greater perspicuity and uniformity in the method of calculation for places on different latitudes.

When the sunlight penetrates into the atmosphere of the earth, it will, for various reasons, be reduced. It is a daily experience that the air is by no means entirely translucent to light. The air contains, in a greater or lesser degree, a number of minute particles of dust and drops of water, which disperse the light in all directions, just as a parallel beam of light is dispersed when it passes through a turbid fluid. The light is also dispersed against the molecules of the air itself, and indeed it is this dispersion that produces the light of the sky. Lord Rayleigh has shown that the short-wave rays are thus dispersed to a greater extent than the long-wave rays, so that the dispersed light contains comparatively more violet and blue rays than the sunlight itself. This is the explanation of the blue color of the sky.

But the air and particles floating in it reduce the light not only by dispersion, but also by sheer absorption. Thus the upper strata of the atmosphere contain ozone, which absorbs part of the ultraviolet solar radiation falling towards the earth, while vapor and carbon dioxide in the lower strata absorb parts of the infrared zone.

The law in accordance with which the light is reduced as it passes through the atmosphere has long been known. It is the same law as that which applies to colored aqueous solutions, glass filters etc., namely Bouguer-Lambert-Beer's law. In accordance with this law the intensity of the light is I, when it has passed a stratum of air of the thickness M,

$$I = I_0\, e^{-aM}$$

or, if we prefer 10 in place of e as base, then

$$I = I_0\, 10^{-\varepsilon M}$$

In these formulas I_0 designates the intensity of the sunlight at the boundary of the atmosphere, and a and ε the so-called absorption or extinction coefficient for the stratum of air in question. Seeing that the air, as above indicated, reduces the intensity of different wave-lengths in different degrees and in different ways, we should, strictly speaking, specify the coefficient for every wave-length and for every cause. On this occasion, however, we do not propose to give such a specified description of the intensity of the sunlight, but will let a and ε, respectively, denote the *average* absorption or extinction coefficient for all wave-lengths and causes. Moreover, seeing that we are not interested in the radiation energy as such, but in the illumination to which it gives rise, we estimate the energy of different wave-lengths in the same way as the eye, thus photometrically, and also include this valuation in the coefficients.

I indicates the intensity of radiation at right angles to the direction of the rays. If we now desire to know the illumination on a horizontal surface, for example, the ground, when the sun is $h°$ above the horizon, we shall easily find that I must be multiplied with the sine h, whereas a multiplication with cosine h gives the illumination on a vertical surface at right angles to the perpendicular plane through the sun (Lambert's cosine law). If the illumination from the sun on the horizontal surface is designated by S, then

$$S = I_0 \sin h \; 10^{-\varepsilon M} \qquad (1)$$

The air mass M penetrated by the rays of the sun is dependent on the height of the sun, too, seeing that the more obliquely the rays fall on the earth, the thicker the stratum of air they have to pass. The least air mass is passed when the sun is at the zenith and $h = 90°$, and this is therefore usually chosen as a unit for the air mass. Bemporad[1] has calculated the thickness of the air mass at different solar altitudes, and his figures are given in every astronomical or meteorological handbook. For the convenience of the reader, however, Bemporad's figures are reproduced in Table I.

TABLE I.—AIR MASS, M, CORRESPONDING TO DIFFERENT SOLAR ALTITUDES (ACCORDING TO BEMPORAD).

Solar Altitude	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°
0°	—	26.96	19.79	15.36	12.44	10.40	8.90	7.77	6.88	6.18
10	5.60	5.12	4.72	4.37	4.08	3.82	3.59	3.39	3.21	3.05
20	2.90	2.77	2.65	2.55	2.45	2.36	2.27	2.20	2.12	2.06
30	2.00	1.94	1.88	1.83	1.78	1.74	1.70	1.66	1.62	1.59
40	1.55	1.52	1.49	1.46	1.44	1.41	1.39	1.37	1.34	1.32
50	1.30	1.28	1.27	1.25	1.24	1.22	1.20	1.19	1.18	1.17
60	1.15	1.14	1.13	1.12	1.11	1.10	1.09	1.09	1.08	1.07
70	1.06	1.06	1.05	1.05	1.04	1.04	1.03	1.03	1.02	1.02
80	1.02	1.01	1.01	1.01	1.01	1.00	1.00	1.00	1.00	1.00
90	1.00									

In order to determine the value of the constants I_0, and ε in the formula (1), we must make use of measurements obtained in practice. Measurements of the radiation of the sunlight are made regularly at a large number of places not only in the United States[2] but also in other countries. Most of them, however, are expressed in energy units and as a rule include also the infrared part of the radiation, whence it is difficult to make use of them for photometric calculations. Thus Kimball[3] has shown that the factor in the conversion of radiant energy from energy to photometric units varies with the color of the light, which in turn is dependent on the height of the sun. The sunlight, as we know, is redder when the sun is nearer the horizon than when it is higher in the sky.

We must, therefore, make use, instead, of directly executed photometric measurements of the daylight illumination. Such measurements have, unfortunately, been made only on a comparatively small scale, but we may mention those made by Kimball and Hand[4] and by Kunerth and Miller.[5] Here, however, we have made use not of these measurements, but of investigations made by Aurén[6] in Sweden and by Lunelund[7] in Finland, which have been conducted (1) with subjective photometers and (2) with photoelectric cells, the spectral sensitiveness of which has been tolerably adjusted by a filter to that of the eye. Seeing that the results obtained by these investigators are based on the average values of an immensely large number of single measurements made in the course of several years and at different places in Scandinavia, they may be considered to be highly reliable. Aurén and Lunelund have moreover adopted an illumination unit which is very well suited for comparative calculations of illumination, namely the total average illumination from sun and sky on clear days at a solar altitude of 45 degrees. This unit, which we, with Aurén, designate by E_s, according to Lunelund, constitutes approximately 77000 lux or about 7150 footcandles.

On the basis of the data given by Lunelund, in particular, which almost exactly tally with those of Aurén, we now determine the value of the constants I_0 and ε in formula (1) to 1.6 E_s units and 0.1, respectively, so that the illumination on a horizontal surface from the solar radiation solely can be written

$$S = 1.6 \sin h \; 10^{-0.1M} = 1.6 \, E_s \sin h \; 10^{-0.1M} \qquad (2)$$

The course of this function is illustrated by the curves in Fig. 1, and the values estimated tally very well with the measurements made by Aurén and Lunelund, which are inserted in the Figure.

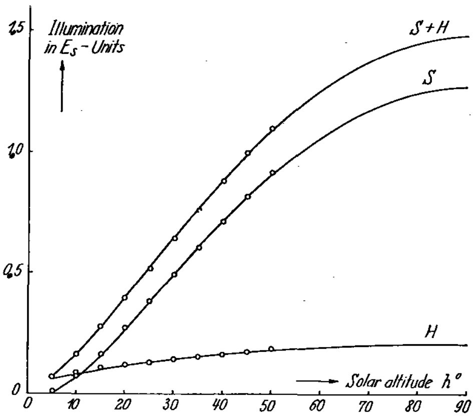

FIG. 1

How then are we to ascertain the solar altitude h on a certain occasion? It depends not only on the time of year and time of day, but also on the latitude of the place, and we can calculate it from the formula

$$\sin h = \sin \varphi \sin \delta + \cos \varphi \cos \delta \cos 15\, t \qquad (3)$$

which is often used by astronomers for calculating, for example, the time of sunrise and sunset. In (3) φ designates the latitude of the place in question, δ the declination of the sun on the day in question and t the number of hours reckoned from true noon. The figures for the declination of the sun and the exact time at a certain place when true noon sets in will be found in the larger almanacs and calendars or in the literature of the subject. In Table II the declination of the sun and the equation of time are given for a number of days in the year.

We have thus collected all the necessary data for calculating the average illumination from the sun in E_s units on a horizontal surface for any day in the year and for any time in the day whatever.[8]

Example. We wish to calculate the illumination from the sun at Stockholm on the 9th July at 4 p.m. Stockholm is on a latitude of 59° 21′, and the difference of time as compared with Swedish standard time is $- 12^m$, and the equation of time on the said day, $+ 5^m$. True noon thus sets in on that day at Stockholm at $12^h - 12^m + 5^m = 11^h\,53^m$. The time from then to 4 p.m. is $4^h\,7^m$ or $4^h.12 = t$. The declination of the sun on the 9th of July, according to Table II, is $+ 22°.5$. Hence according to formula (3)

TABLE II—DECLINATION OF THE SUN AND EQUATION OF TIME.

Date		Declination	Equation of Time	Date		Declination	Equation of Time
January	1	−23°.1	+3^m	July	9	+22°.5	+5^m
	10	−22.1	+7		19	+21.0	+6
	20	−20.3	+11		29	+18.9	+6
	30	−17.8	+13	August	8	+16.4	+6
February	9	−14.9	+14		18	+13.3	+4
	19	−11.5	+14		28	+10.0	+1
March	1	−7.8	+13	September	7	+6.3	−2
	11	−4.0	+10		17	+2.5	−5
	21	0.0	+7		27	−1.4	−9
	31	+3.9	+4	October	7	−5.2	−12
April	10	+7.7	+2		17	−9.0	−14
	20	+11.3	−1		27	−12.6	−16
	30	+14.6	−3	November	6	−15.8	−16
May	10	+17.5	−4		16	−18.6	−15
	20	+19.8	−4		26	−20.8	−13
	30	+21.7	−3	December	6	−22.4	−9
June	9	+22.9	−1		16	−23.3	−5
	22	+23.5	+2		23	−23.5	−1
	30	+23.2	+3		30	−23.2	+2

$$\sin h = \sin 59°21' \sin 22°.5 + \cos 59°21' \cos 22°.5 \cos 15 \times 4.12$$
$$\sin h = 0.329 + 0.223$$
$$\sin h = 0.552$$
$$h = 33°.5$$

From Table I we find that the figure for the air mass M corresponding to the solar altitude 33°.5 is 1.80. Thus, according to the formula (2)

$$S = 1.6 \cdot 0.552 \cdot 10^{-0.1 \cdot 1.80}$$
$$\log S = 0.204 + 0.742 - 1 - 0.180 = 0.766 - 1$$
$$S = 0.583\ E_s \text{ units.}$$

The figures for the illumination from the sun given by the formula (2) relate in the first place to an atmosphere similar to that which prevails on an average in Scandinavia and Finland (55–70° north latitude). It is however, known[9] that the air in general is clearer and more transparent the farther north one gets. This signifies that the illumination from the sun is less intense at more southerly latitudes than at more northerly even at the same solar altitude. In southern zones we should therefore strictly speaking, reckon with a somewhat larger coefficient of extinction in (2) and with a somewhat lower figure for I_0 than in northern regions

If we study the data given by Kunerth and Miller,[10] obtained at a latitude of 42°, we shall thus find that they conform to the same law as that which is expressed by the formula (1) though the figure for ε will be 0.4 and for I_0 1.23. In this case Kunerth and Miller's figures in footcandles are recalculated in Aurén's E_s unit. It should, however, be pointed out that these figures for ε and I_0 are not solely determined by the southerly latitude, but also by local atmospheric conditions at the place of observation in question (Ames, Iowa).

Illumination from the Sky

The radiation from the sky is less variable according to the solar altitude than the sunlight itself, and, in view of its complicated nature, it is very difficult theoretically to calculate its intensity. We will not, therefore, discuss here the attempts that have been made so to do, but will merely refer to the calculation[11] that the illumination H in E_s units on a horizontal surface from a clear sky, with fairly good approximation, can be illustrated by the formula

$$H = 0.211 \sin^{0.5} h = 0.211\ E_s \sqrt{\sin h} \qquad (4)$$

This formula can be used for greater solar altitudes than about 3°; for $h < 3°$ too small figures for the skylight are obtained.

Formula (4) is based on Lunelund's measurements,[12] but Kunerth and Miller's data for the skylight[13] will be found to conform to the same law, although the constant factor figures at 0.14 and the exponent at 0.65. But, seeing that the latter's observations were made at a single place and during a relatively short space of time, whereas Lunelund made extensive measurements at a large number of places and for a long series of years, we seem to be warranted in regarding the figures in formula (4) as of more general validity.

The curve H in Fig. 1 illustrates the variation in the illumination from the sky according to the height of the sun. The figures inserted are Lunelund's measurements. The total illumination from sun and sky is obtained by the addition of S and H and is represented by the curve $S + H$ in the figure.

The radiation from the sky is fairly constant during the greater part of the day, whereas the intensity of the sunlight falls relatively rapidly according as the sun descends. It is not until the approach of sunset that the skylight begins to show a marked diminution, but even after the sun has passed below the horizon it has a noteworthy intensity (twilight). Owing to the fact that the sunlight diminishes so much more rapidly than the skylight, we find that at a certain low height of the sun the sunlight and skylight have the same intensity. Thus the proportional contribution from the sky to the total illumination on the ground increases more and more according as the sun sets.

Illumination from a Cloudy Sky

Clouds change the illumination in different degrees according to their appearance, extent and density. Sometimes clouds may actually intensify the illumination; for example, light, white clouds, which, owing to the reflection of the solar radiation, increase the illumination in the shadows. In most cases, however, clouds reduce the solar radiation by absorption and dispersion, but the reduction of the sunlight and of the skylight is different, in that the parallel sunlight is most reduced by clouds in its path. The light which reaches through the clouds down to the surface of the earth may now be regarded, as the investigators[14] have shown, as the sum-total of the sunlight reduced by a certain fraction, and the light from the clear sky above the clouds, reduced by another fraction. The illumination on a horizontal surface from a clouded sky can thus as a rule be written:

$$W = x\,S + y\,H,$$

where x and y represent the fractions by which the sunlight and skylight, respectively, are reduced.

Lunelund, in his above-mentioned investigation,[15] has reported measurements of illumination made under different conditions of cloud at different solar altitudes in E_s units, and his results, as the authors have shown, may well be expressed by the following formulas:

(a) illumination in the shade from a sky with light, white clouds and a bare sun:

$$W_a = 0.08\,S + 1.02\,H \qquad (5a)$$

(b) total illumination when the sun is screened by a thin film of cloud:

$$W_b = 0.35\,S + 0.89\,H \qquad (5b)$$

(c) total illumination from a clouded sky:

$$W_c = 0.26\,S + 0.54\,H \qquad (5c)$$

The formula (5a) indicates that light clouds in the sky reflect about 8 per cent of the sunlight into the shadows, and that the skylight, owing to a somewhat increased dispersion of the solar radiation, is magnified by 2 per cent.

TABLE III—RELATIVE ILLUMINATION IN TERMS OF E_s UNITS AT DIFFERENT HEIGHTS OF THE SUN ON A HORIZONTAL SURFACE FROM SUN; S, FROM CLEAR SKY, H; AND FROM BOTH SUN AND SKY, $S + H$, CALCULATED FROM EQ. (2) AND (4).

Solar altitude	S	H	$S + H$
5°	0.013	0.062	0.075
10	0.077	0.088	0.165
15	0.172	0.107	0.279
20	0.280	0.124	0.404
25	0.393	0.137	0.530
30	0.506	0.150	0.656
35	0.615	0.160	0.775
40	0.719	0.169	0.888
45	0.819	0.178	0.997
50	0.908	0.185	1.093
55	0.989	0.191	1.180
60	1.064	0.197	1.261
65	1.125	0.201	1.326
70	1.178	0.205	1.383
75	1.216	0.208	1.424
80	1.245	0.209	1.454
85	1.265	0.211	1.476
90	1.271	0.211	1.482

It will be seen from formula (5b) that the solar radiation is reduced down to about one-third by thin, light clouds, whereas the skylight is merely changed from 100 to about 90 per cent.

Formula (5c) states that the sunlight, owing to a clouded sky, is reduced to roughly 25 per cent and the skylight to roughly 50 per cent of their respective values in a clear, cloudless sky.

The formulas now mentioned, (2), (4), (5a), (5b) and (5c), in combination with (3), permit the calculation of the average illumination on a horizontal surface in E_s units from sun and sky, firstly for a clear sky and secondly for various combinations of clouds. In Table III the illumination is calculated from sun, from clear sky and from both sun and sky at different solar altitudes.

The above figures for illumination relate to a relatively clear and translucent air, which is thus neither extremely clear, as often in mountainous regions, nor markedly filled with vapors, as in industrial districts or in places with other unfavorable atmospheric conditions.

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