

Mathematical and Physical Bases for Incandescent Lamp Exponents

By DAVID D. VAN HORN

IT HAS been customary^{1, 2, 3} to describe the relationship between measurable properties or parameters of an incandescent lamp by means of equations involving "lamp exponents." Thus, for example, the relationship between the lumen output and the voltage applied to a lamp is given by

$$\left(\frac{\text{lumens}}{\text{LUMENS}} \right) = \left(\frac{\text{volts}}{\text{VOLTS}} \right)^k, \quad (1)$$

where the lower case quantities are variables and the upper case quantities are the normal rated values of these variables. k is the exponent for this particular pair of parameters. The value of k used in Equation (1) depends upon the particular lamp type and also upon the range of voltage considered. In other words, k may not be a "constant" even for a single lamp, but depends upon voltage range and the definition of "normal rated volts." Table I lists the various lamp parameters and the symbols which will be used for them. Mathematical equations, such as Equation (1), connecting these incandescent lamp operating characteristics have two uses:

1. As computational tools for extrapolating from known values to unknown values of the parameters, and

2. As means of discovering physical principles involved in the operational behavior of lamps.

The significance of Equation (1) is made clearer if we look at other equations which are mathematically equivalent. Equation (1) expresses a functional relationship between lumens and volts in a form which is convenient for purposes of calculation. Other equivalent forms are:

A paper presented at the National Technical Conference of the Illuminating Engineering Society, August 30 to September 3, 1964, Miami Beach, Fla. AUTHOR: General Electric Co., Large Lamp Dept., Nela Park, Cleveland, Ohio. Accepted by the Papers Committee as a Transaction of the IES.

Mathematical equations connecting incandescent lamp operating characteristics have two uses: (1) as computational tools, and (2) as means of discovering physical principles involved in the operational behavior of lamps. The only equations which have been used to any great extent for the first of these purposes are those involving the so-called lamp exponents. The physical basis for the behavior of incandescent lamps is found in the variation of various properties of tungsten with temperature. A simple theoretical derivation of the lamp exponents has been developed. The simplified theory as presented gives results that are close enough to the observed behavior of lamps that corrections to the theory for end effects, blackbody effect of helical coils, etc., should be possible without the need of excessive elaborations.

$$(\text{lumens}) = (\text{constant}) \times (\text{volts})^k \quad (2)$$

and

$$\log(\text{lumens}) = (\text{constant}) + k \log(\text{volts}) \quad (3)$$

Thus the use of exponents as exemplified in Equation (1) is more fully described by the following statement: Within small ranges of their rated values, any

Table I—Lamp Parameters

Electrical:	V	Volts
	I	Current
	W	Watts
	R	Resistance
Optical:	L	Lumens
	E	Efficacy (lumens/watt)
"Chemical":	A	Life

two parameters of an incandescent lamp vary in such a way that there is a linear relationship between their logarithms.

A still more concise expression is:

$$k = \frac{d \log (\text{lumens})}{d \log (\text{volts})} \left[= \frac{V dL}{L dV} \right] \quad (4)$$

This is the most convenient of all the equivalent forms, since it not only expresses the relationship between lumens and volts most concisely, even for cases where *k* itself may depend upon the voltage, but also points out most clearly the interrelationships among the various exponents. For example, if we take $g = (d \log E)/(d \log V)$ and $t = (d \log I)/(d \log V)$, then we can find *g* in terms of *k* and *t*.

Since $E = \frac{L}{W} = \frac{L}{IV}$, then $\log E = \log L - \log I - \log V$. Therefore,

$$\frac{d \log E}{d \log V} = \frac{d \log L}{d \log V} - \frac{d \log I}{d \log V} - \frac{d \log V}{d \log V} \quad \text{or} \\ g = k - t - 1$$

Table II lists all of the exponents in common use and their definitions. It also gives the relationship of these exponents to the first three. The reason for choosing *d*, *k*, and *t* as the basic exponents is the fact that life, lumen output, and current are physical quantities most easily measured as functions of the voltage applied to lamps. (Unfortunately, life can only be measured statistically in many lamps, since each lamp has but a single life, while lumens and current may be measured on a single lamp over the whole range of voltages). Note that, in addition to voltage which is taken as the independent variable, there is an exponent in this basic set related to each of the three groups of parameters (electrical, optical, and chemical) of Table I.

Table II—Definitions and Interrelations of Lamp Exponents

DEFINITION	EXPONENT	EQUIVALENT
$-(d \log \Lambda)/(d \log V)$	<i>d</i>	
$(d \log L)/(d \log V)$	<i>k</i>	
$(d \log I)/(d \log V)$	<i>t</i>	
$(d \log W)/(d \log VP)$	<i>n</i>	<i>t</i> + 1
$(d \log E)/(d \log V)$	<i>g</i>	<i>k</i> - <i>t</i> - 1
$(d \log E)/(d \log L)$	<i>f</i>	$(k - t - 1)/k$
$(d \log E)/(d \log I)$	<i>j</i>	$(k - t - 1)/t$
$-(d \log \Lambda)/(d \log L)$	<i>a</i>	<i>d</i> / <i>k</i>
$-(d \log \Lambda)/(d \log E)$	<i>b</i>	<i>d</i> / $(k - t - 1)$
$-(d \log L)/(d \log I)$	<i>u</i>	<i>d</i> / <i>t</i>
$(d \log L)/(d \log E)$	<i>h</i>	$k/(k - t - 1)$
$(d \log L)/(d \log W)$	<i>s</i>	$k/(t + 1)$
$(d \log L)/(d \log I)$	<i>y</i>	<i>k</i> / <i>t</i>
$(d \log L)/(d \log R)$	<i>z</i>	$k/(I - t)$

As mentioned previously, one purpose of using these exponents is for purposes of calculation. In this regard, one is interested both in accuracy and in convenience. Equations (1) or (2) are very easy to use, but they suffer from two major limitations. In the first place, as mentioned above, the value of any particular exponent is not the same for all lamps but differs from lamp type to lamp type. Lacking a theoretical basis for determining the exponents, the number used must be obtained empirically. Additionally, the exponent may only be used safely over the same range of voltage as that for which they were determined. In any case the accuracy of the resulting calculations depends on how "constant" the exponent is in the voltage range covered.

The accuracy of the calculations may be increased by expressing the functional relationship between parameters in a more complicated form than in Equations (1) to (4). For example,^{4,5} the National Bureau of Standards has determined constants for various lamps in equations of the form:

$$\log L = A' + B' \log V + C' (\log V)^2 \quad (5)$$

Because of the quadratic term, such an equation can, for given values of *A'*, *B'*, and *C'*, describe the behavior of a lamp over a wider range of voltage than is possible with the corresponding linear equation [Equation (3)]. It is obvious, however, that it is less convenient to use, and, most important, it becomes extremely difficult to express the interrelationships between various sets of constants. Hence the possible gain in accuracy is offset by the inconveniences introduced. Furthermore, the more complete representation of the functional relationship between parameters given by Equation (5) is also obtainable from Equation (4) if we consider the exponent not as a constant but as a variable. Hence a completely accurate description is given by,

$$\frac{d \log L}{d \log V} = k(V) \quad (4')$$

where we now write the exponent as a function of the independent variable *V*.

The difficulty with the exponent equations as normally used lies in the apparent lack of any rational basis for the observed variation of the exponents with voltage or the observed differences in the exponents among different lamp types.

What will now be shown is that from the physical properties of tungsten it is possible to explain much of what is observed about the behavior of the exponents. The most important step in this analysis is the adoption of temperatures, rather than voltage, as the fundamental independent variable of a lamp upon which all of its other properties depend. Table III is a list of the temperature dependent properties of tungsten which are necessary for the development of a physical basis for the lamp exponents.

Table III—Properties of Tungsten

PROPERTY	SYMBOL	UNITS
Electrical resistivity	ρ	ohm-cm.
Total radiant emittance	η	watts/cm ²
Radiant efficacy	ϵ	lumens/watt
Evaporation rate	m	grams/cm ² -second

We are, for the moment, not interested in the absolute values of these properties, but only in the rate at which they vary with temperature. These variations may also be expressed in exponential form, and, in fact, have been so expressed by many sources.^{1, 3, 5, 7, 8} Table IV gives the fundamental exponents to be used.

Now, in order to simplify the theory, two assumptions will be made:

1. We will neglect the fact that the temperature of a filament is not uniform; *i.e.*, we will neglect end effects; and

2. We will neglect the effect of coiling on the effective emissivity of the filament.

Although these assumptions may seem to be a great over-simplification, yet we will find the results to be quite reasonable. Since the end losses of a lamp are on the order of one or two per cent, the first assumption seems well justified. The neglect of coiling affects the results but little, probably because, although the emissivity is increased due to the black-body effect of coiling, the effective radiation area is decreased.

We now take the following geometric quantities as being descriptive of a vacuum-lamp filament:

L' = length of filament wire,

d' = diameter of filament wire,

A' = effective radiating area of filament.

The electrical characteristic of the filament is, therefore, given by:

$$\frac{V}{I} = R = \rho \frac{\pi d'^2}{4L'} \quad (6)$$

Since we are neglecting the end effects, all of the electrical energy must be radiated as thermal energy. Hence

$$VI = W = \eta A' \quad (7)$$

It follows that the luminous output of the lamp is given by:

$$L = \epsilon \eta A' \quad (8)$$

If we assume that the life of the lamp is inversely proportional to the evaporation rate of the filament,

$$\Lambda = (\text{constant})' x \frac{1}{m} \quad (9)$$

Table V—Fundamental Exponents

$$(d \log \rho)/(d \log T) = c$$

$$(d \log \eta)/(d \log T) = x$$

$$(d \log \epsilon)/(d \log T) = q$$

$$(d \log m)/(d \log T) = p$$

(T = absolute temperature in degrees K)

Note that all of the primed terms are geometric constants and will be independent of temperature.

If we now take the logarithmic derivative with respect to temperature of each of the above four equations, we get

$$\frac{d \log V}{d \log T} - \frac{d \log I}{d \log T} = \frac{d \log \rho}{d \log T} = c \quad (10)$$

$$\frac{d \log V}{d \log T} + \frac{d \log I}{d \log T} = \frac{d \log \eta}{d \log T} = x \quad (11)$$

$$\frac{d \log L}{d \log T} = \frac{d \log \epsilon}{d \log T} + \frac{d \log \eta}{d \log T} = q + x \quad (12)$$

and

$$\frac{d \log \Lambda}{d \log T} = - \frac{d \log m}{d \log T} = -p \quad (13)$$

From Equations (10) and (11), we get

$$\frac{d \log V}{d \log T} = (c + x)/2 \quad (14)$$

and

$$\frac{d \log I}{d \log T} = (x - c)/2 \quad (15)$$

Now dividing Equations (12), (13) and (15) by Equation (14) (that is, eliminating the temperature variable between the equations), we obtain finally,

$$\frac{d \log I}{d \log V} = t = \frac{x - c}{x + c} \quad (16)$$

$$\frac{d \log L}{d \log V} = k = \frac{2(q + x)}{c + x} \quad (17)$$

and

$$- \frac{d \log \Lambda}{d \log T} = d = \frac{2p}{c + x} \quad (18)$$

Note that we have succeeded in determining all of the lamp exponents in terms of the physical properties of tungsten as expressed by the fundamental exponents of Table IV.

The above analysis was made for vacuum lamps. The only change which must be made for gas-filled lamps is the inclusion of the convective cooling of the filament by the gas.

Unlike the end effect, this cooling may be of the order of ten to 20 per cent of the total energy input to the lamp. For simplicity, we will assume that this cooling may be expressed as a constant (α) times the temperature, times the surface area. Equation (7) becomes, therefore,

$$VI = W = (\eta + \alpha T) A' \quad (7')$$

Equation (8) remains unchanged, since the lumen output is proportional not to the total energy lost by the filament, but only to the radiant energy lost. Equation (11) now becomes

$$\frac{d \log V}{d \log T} + \frac{d \log I}{d \log T} = \frac{d \log (\eta + \alpha T)}{d \log T} = x' \quad (11')$$

To evaluate this expression, we let G be the fraction of the energy input lost to the gas. Thus

$$G = \frac{\alpha T}{\eta + \alpha T} = \frac{\alpha T/\eta}{1 + \alpha T/\eta} \quad (19)$$

This gives $\frac{\alpha T}{\eta} = \frac{G}{(1-G)}$ and $1 + \frac{\alpha T}{\eta} = \frac{1}{1-G}$

Now

$$\begin{aligned} x' &= \frac{T}{\eta + \alpha T} \frac{d(\eta + \alpha T)}{dT} \\ &= \frac{T}{\eta + \alpha T} \left(\frac{d\eta}{dT} + \alpha \right) \text{ or,} \\ x' &= \frac{\eta}{\eta + \alpha T} \frac{T}{\eta} \left(\frac{d\eta}{dT} + \alpha \right) \end{aligned}$$

Therefore,

$$x' = \frac{1}{1 + \alpha T/\eta} \left(\frac{d \log \eta}{d \log T} + \frac{\alpha T}{\eta} \right)$$

or finally

$$x' = (1 - G) x + G \quad (20)$$

Using Equation (11') with Equation (10) gives

$$\frac{d \log V}{d \log T} = (c + x')/2 \quad (14')$$

and

$$\frac{d \log I}{d \log T} = (x' - c)/2 \quad (15')$$

Consequently,

$$t = \frac{x' - c}{x' + c} \quad (16')$$

$$k = \frac{2(q + x)}{c + x'} \quad (17')$$

$$d = \frac{2p}{c + x'} \quad (18')$$

for the exponents for the lamps that are gas filled.

Now in order to test the results of the theory developed above, we must have numerical values for the fundamental exponents of Table IV. Fig. 1 gives the temperature dependence of these exponents which have been calculated from various sources. The exponents for resistivity, total radiant emittance, and efficacy (c , x and q , respectively) were obtained from the tables of Forsythe and Adams,³ but were corrected for the change (1948) in the international temperature scale.⁹ (The Smithsonian Tables⁸ recognized the change in the temperature scale, but the corrections were made in the wrong direction.) Two sources have been used for the calculation of p , the exponent associated with the evaporation rate of tungsten. The solid line of Fig. 1 for $p/10$ was calculated from the data of Jones, Langmuir and Mackay,¹⁰ while the dashed line was calculated from Reimann.¹¹ In both cases, corrections were made to the 1948 temperature scale, assuming that the originals were consistent with the scale of 1927. This may not be strictly true for the data of Jones, Langmuir and Mackay, and may in fact account for a large share of the difference between the two sets of data.

Now from Equations 16, 17, 18, 16', 17', and 18'

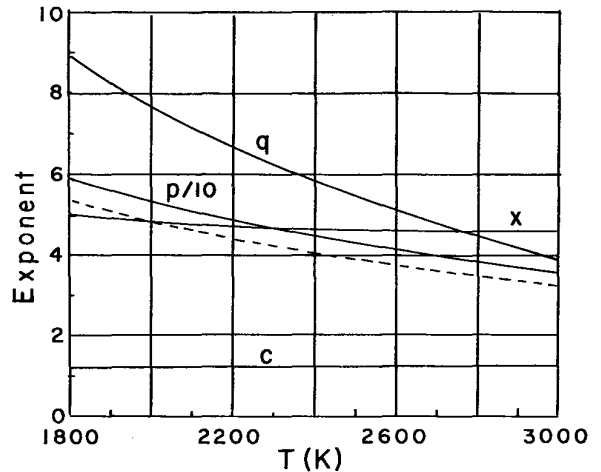


Figure 1. Fundamental Exponents: $c = (d \log \rho)/(d \log T)$; $x = (d \log \eta)/(d \log T)$; $q = (d \log \epsilon)/(d \log T)$; $p = (d \log m)/(d \log T)$; $p/10$ — from Reference 10; $p/10$ — from Reference 11.

we can calculate values of the three basic lamp exponents for various temperatures and for both vacuum and gas-filled lamps. In calculating the exponents for gas-filled lamps, a value of one-eighth was assigned to G , the fraction of the energy lost by gas convection. This is approximately equal to the average values found for modern 60- and 100-watt lamps.¹² The results are shown in Figs. 2, 3 and 4. Also shown (open circles) are the values of the exponents taken from the *IES Lighting Handbook*.² The operating temperature of a typical vacuum lamp was taken as 2560 C and that of a gas-filled lamp as 2770 C. Admittedly, these figures may not correspond exactly to any given lamp, but the same is true of the published values of the exponents so more refinement would not be warranted. We can obtain from the graphs several significant observations:

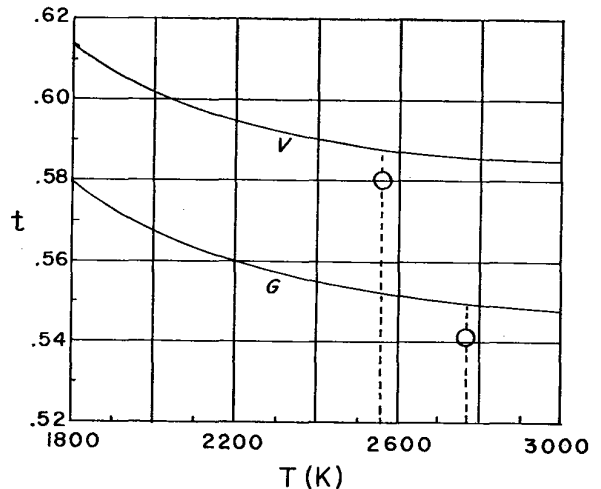


Figure 2. Variation in t exponent. V—vacuum lamp, G—gas-filled lamp.

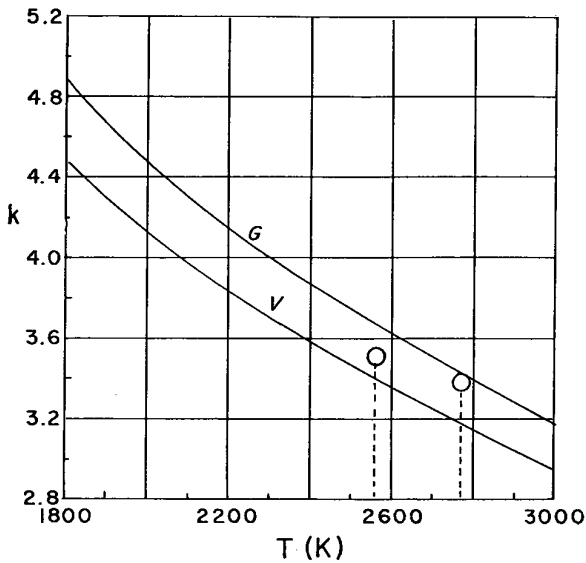


Figure 3. Variation in k exponent. V—vacuum lamp, G—gas-filled lamp.

1. The t and k exponents are predicted with a two or three per cent error.
2. The difference between the t exponents for vacuum and gas-filled lamps is much greater than can be accounted for by the effect of gas losses on the exponent.
3. The difference in d exponents is less than the temperature difference would indicate, but is accounted for by the gas loss effect.
4. Considering the closeness of agreement between theory and observations for t and k , it appears that

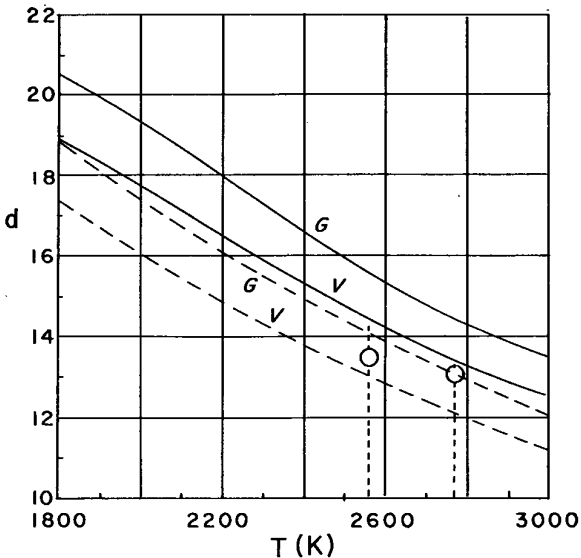


Figure 4. Variation in d exponent. V—vacuum lamp, G—gas-filled lamp. — Using data of Reference 10; ----- using data of Reference 11.

the vaporization data of Riemann are much better than those of Jones, Langmuir and Mackay, since Riemann's data are much more consistent with the observed values of d .

Let us finally examine some of the advantages and shortcomings of the above development. Obviously the theory should be tested more fully by using the data obtained from specific lamp types and by considering the ways in which G , the gas loss of gas-filled lamps, might vary with temperature and with filament design. Further refinements of the theory would include the end effects of the filaments and how they vary with filament design and temperature. We have seen, however, that even making some bold simplifying assumptions, we have succeeded in explaining the major portion of the variations of lamp exponents. Hence we need make only small corrections in the derivation of the exponents. It does not seem unreasonable, therefore, to expect that the corrected theory would give a very useful tool for predicting lamp exponents, but perhaps more important, give a much better tool than we now possess for computing filament design from first principles. Hence the use of lamp exponents is much more than a mathematical convenience, but is also a framework upon which to develop a fuller understanding of the physical principles of the incandescent lamp.

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DISCUSSION

VICTOR BEISSWINGER:* I am disappointed that Dr. Van Horn has avoided discussion of life characteristics because a statistical sample would be required. I represent the viewpoint of the large-lamp user. Los Angeles has a statistical sample

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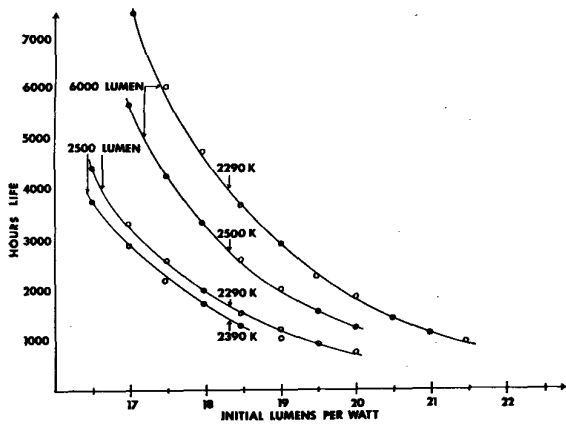


Figure A. Constant color temperature curves 6000 lumen, 2500 lumen—6.6 amps. (Beisswinger discussion.)

of 115,000 incandescent lamps burning every night in the street light system. The observed differences in the luminous characteristics of incandescent lamps of various design and manufacture are not important, but the differences in life characteristics greatly affect the annual cost of operating the street light system.

I do not question Dr. Van Horn's mathematical approach to the problem, but I do question his assumption that the effect of coiling on the effective emissivity of the filament will be neglected. I also question the omission of color temperature entirely.

I refer you to page 1-11 of the *IES Lighting Handbook*, which says, "The radiation characteristics of a blackbody of an unknown area may be specified with the aid of the equations on page 1-9 by fixing only two quantities: the magnitude of the radiation at any given wavelength and the absolute temperature. The same type of specification may be used with reasonable accuracy for tungsten filaments and other incandescent sources. However, the temperature used in the case of selective radiators is not that of the filament but a value called the color temperature."

Using this method of specifying total radiation of tungsten filaments, a test was made in 1963 by the Los Angeles Department of Water and Power Testing Laboratories that was unique in this respect. The life curve was accurately predicted in advance of the test. The results are shown in Fig. B.

A similar test is being conducted on six additional samples that will furnish data that I hope to incorporate in a paper for the 1965 Technical Conference.

R. E. LEVIN:* I commend the author for his clear analysis and for pointing out that temperature is a better independent variable than voltage for specifying lamp characteristics.

The assumption that the coiling of a filament does not affect effective emissivity is a valid approximation in many cases. However, as the author points out, a balance exists between the "blackbody effect" and the effective radiating area. Incandescent lamps have been measured with an effective emissivity as much as 40 per cent greater than that of the base material. These measurements were performed in the short wavelength infrared and indicate values that may be attained in the visible spectrum.

Another advantage is accrued when lamp properties are

*Sylvania Lighting Products Inc., Danvers, Mass.

expressed as a function of temperature rather than voltage. For lamps within properly defined categories, electrical and luminous characteristics are essentially the same for all lamps and valid over a large range of operating conditions providing normalization to temperature is used. As a brief example, Fig. C shows normalized lamp lumens for lamps from a class of concentrated filament quartz-iodine lamps. Here the temperature of the central part was used, *i.e.*, end effects were neglected. To illustrate the range covered, the four lamps for Fig. C were designed for the following rating: 250 watt, 10 hour; 250 watt, 2000 hour; 1000 watt, 25 hour and 1500 watt, 2000 hour.

DAVID D. VAN HORN:* I must agree with Mr. Beisswinger that I have not discussed life characteristics per se, nor have I treated any other characteristic by itself, since such discussions are entirely outside the intended scope of this paper. In the terms of the paper, I have dealt only with the exponent in relationships of the type represented by Equation (2) of the paper. I believe that Mr. Beisswinger is concerned primarily with the nature of the "constant" terms in such equations. I am convinced that differences in performance among individuals of the same lamp type will depend upon minor differences in filament geometries, gas pressures, and of course, even on the operating position of the lamp. The lamp exponents that have been discussed in the paper deal primarily with the changes in characteristics of an individual lamp related to changes in operating voltage. Nonetheless, there is valuable information about the statistical behavior of large samples of lamps, which may be obtained from the lamp exponents. Suppose, for example, that variations between individuals were caused by variations in filament geometry (length, wire diameter, etc.) which in turn would cause variations in the filament resistance and hence the filament operating temperature. Assuming these lamps are gas filled and operate at 2770 K, we find from Fig. 1 of the paper that the exponent relating efficacy to temperature (q) is 4.55 and the corresponding exponent for evaporation rate, and hence life (p), is 38.5. Therefore

$$\frac{d \log T}{d \log \epsilon} = \frac{p}{q} = \frac{38.5}{4.55} = 8.46$$

The laws of statistics lead to the result that the dispersion of life in the sample as measured by per cent sigma, will be 8.46 times as large as the per cent sigma of the dispersion

* Author.

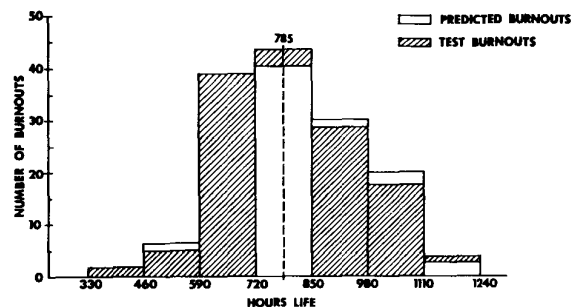


Figure B. Life prediction of 6000-lumen series lamps (accelerated life test). The predicted median was 785 hours; the test median, 791. The predicted mean was 804 hours; test mean, 809. (Beisswinger discussion.)

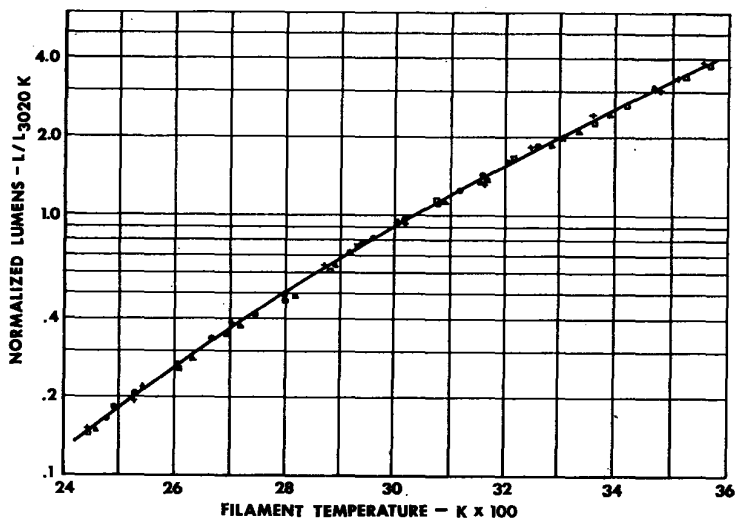


Figure C. (Levin discussion).

in efficacy. Any other effects, such as wire inhomogeneity, that would effect life without affecting efficacy, would increase the dispersion in life still further. In this way a better knowledge of lamp exponents may be valuable in the design and analysis of experiments concerning distributions of lamp characteristics.

I have not used color temperature in my development of exponents since I have attempted to rationalize the behavior of lamps on the basis of the physical properties of tungsten. The true temperature is the physical quantity that determines the filament's resistivity, emissivity, efficacy, etc. Filaments may have the same true temperature but will have differing

color temperatures if the filaments differ in their geometry. I am unable to comment on Mr. Beisswinger's figures since I do not understand how the initial lumens per watt of the 6.6-amp street series lamps could be varied as shown without a simultaneous variation in the operating color temperature.

I am indeed grateful to Mr. Levin for presenting his results. His data point out quite clearly not only that the true temperature is indeed a proper independent variable upon which to base a study of the variation of lamp characteristics but also the fact that differences in filament geometries have little or no effect on the manner in which the operating characteristics change with temperature.